Math 10A with Professor Stankova
Worksheet, Discussion #25; Monday, 10/23/2017
GSI name: Roy Zhao

Improper Integrals

Example

1. Let $\alpha \in \mathbb{R}$. Calculate $\displaystyle\int_1^\infty \frac{1}{x^\alpha} dx$.

> **Solution:** We can rewrite this as $\frac{1}{x^\alpha} = x^{-\alpha}$. For $\alpha \neq 1$, we know that
> $$\frac{d}{dx}\frac{x^{1-\alpha}}{1-\alpha} = \frac{1}{x^\alpha},$$
> and hence we have that
> $$\int_1^\infty \frac{1}{x^\alpha} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^\alpha} dx = \lim_{t\to\infty} \left[\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right].$$
> Now if $1 - \alpha > 0$, or $\alpha < 1$, then this limit is infinite. If $\alpha > 1$, then the first term disappears and we just get $\frac{1}{\alpha-1}$. Now if $alpha = 1$, then this is
> $$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} [\ln t - \ln 1] = \infty.$$
> Therefore, we have that
> $$\int_1^\infty \frac{1}{x^\alpha} dx = \begin{cases} \infty & \alpha \leq 1 \\ \frac{1}{\alpha-1} & \alpha > 1. \end{cases}$$

Problems

2. **TRUE** False It is possible for the integral $\displaystyle\int_1^\infty f(x)$ to be neither a finite number nor infinity.

> **Solution:** We showed that the integral $\int_0^\infty \cos(x) dx$ doesn't exist at all in class.

3. True **FALSE** By the above example, we know that $\int_0^\infty \frac{1}{x^3}dx$ converges.

> **Solution:** We only showed the result where the bottom limit is 1. This integral actually diverges.

4. True **FALSE** If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)dx$ converges.

> **Solution:** Counter example is $f(x) = \frac{1}{x}$.

5. Calculate $\int_3^\infty \frac{1}{x\ln(x)}$.

> **Solution:** We have that
>
> $$\int_3^\infty \frac{1}{x\ln x} = \lim_{t\to\infty} \int_3^t \frac{1}{x\ln x} = \lim_{t\to\infty} \ln(\ln x)|_3^t$$
>
> $$= \lim_{t\to\infty}[\ln(\ln(\infty)) - \ln(\ln 3)] = \infty.$$

6. Calculate $\int_1^\infty e^{-5x}dx$.

> **Solution:** We have that
>
> $$\int_1^\infty e^{-5x}dx = \lim_{t\to\infty} \int_1^t e^{-5x}dx = \lim_{t\to\infty} \frac{e^{-5x}}{-5}|_1^t$$
>
> $$= \lim_{t\to\infty}[\frac{e^{-5t}}{-5} + \frac{e^{-5\cdot 1}}{5} = \frac{e^{-5}}{5}.$$

7. Calculate $\int_1^\infty \frac{x}{\sqrt{x^2+1}}dx$.

> **Solution:** We have that
>
> $$\int_1^\infty \frac{x}{\sqrt{x^2+1}}dx = \lim_{t\to\infty} \int_1^t \frac{x}{\sqrt{x^2+1}}dx = \lim_{t\to\infty} \sqrt{x^2+1}|_1^t$$
>
> $$= \lim_{t\to\infty}[\sqrt{t^2+1} - \sqrt{2}] = \infty.$$

8. Calculate $\displaystyle\int_0^\infty \frac{1}{1+x^2}dx$.

Solution: We have that

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\arctan(x)_0^t = \lim_{t\to\infty}\arctan(t) = \frac{\pi}{2}.$$

9. Calculate $\displaystyle\int_1^\infty xe^{-2x}dx$.

Solution: First we calculate that

$$\int xe^{-2x}dx = x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}dx = \frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C,$$

by integration by parts. Thus, we have that

$$\int_1^\infty xe^{-2x}dx = \lim_{t\to\infty}\int_1^t xe^{-2x}dx = \lim_{t\to\infty}\left[\frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right|_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{e^{-2}}{2} + \frac{e^{-2}}{4}\right] = \frac{e^{-2}}{2} + \frac{e^{-2}}{4}.$$

This is because we can use L'Hopital's rule to calculate that $\lim_{t\to\infty} te^{-t} = 0$.

10. Calculate $\displaystyle\int_1^\infty \frac{2x}{1+x^2}dx$.

Solution: We have that

$$\int_1^\infty \frac{2x}{1+x^2}dx = \lim_{t\to\infty}\int_1^t \frac{2x}{1+x^2}dx = \lim_{t\to\infty}\ln(1+x^2)|_1^t = \infty.$$

Convergent/Divergent Integrals

Example

11. Does $\displaystyle\int_0^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}}dx$ converge?

Solution: First we know that $|\arctan(x)| \leq \frac{\pi}{2}$ and so $\arctan^2(x) \leq \frac{\pi^2}{4}$. Also, we know that $1 + x^4 \geq x^4$ and so $\sqrt{1+x^4} \geq \sqrt{x^4} = x^2$ and $\frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$. Thus

$$0 \leq \int_1^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}} dx \leq \int_1^\infty \frac{\pi^2/4}{x^2} dx = \frac{\pi^2}{4}.$$

So this integral converges.

Then for the remaining part, we know that $1 + x^4 \geq 1$ and so $\sqrt{1+x^4} \geq 1$ and $\frac{1}{\sqrt{1+x^4}} \leq 1$ and so

$$\int_0^1 \frac{\arctan^2(x)}{\sqrt{1+x^4}} dx \leq \int_0^1 \frac{\pi^2/4}{1} dx = \frac{\pi^2}{4}.$$

Therefore

$$0 \leq \int_0^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}} dx \leq \frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{\pi^2}{2}.$$

This integral converges.

Problems

12. True **FALSE** If $a < b$ then $ac < bc$.

Solution: We have that $1 < 2$ but $-1 \not< -2$.

13. True **FALSE** If $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

Solution: Take $-1 < 2$ but $-1 \not> \frac{1}{2}$.

14. True **FALSE** If we can find a function g such that $0 \leq f \leq g$, then $\int_1^\infty f(x)dx$ converges.

Solution: In order to say the integral of f converges, we need to show that the integral of g converges as well.

15. Does $\int_3^\infty \frac{1}{\sqrt{x}\ln(x)}$ converge?

Solution: We know that for $x \geq 3$ that $x \geq \sqrt{x}$ and so $x\ln(x) \geq \sqrt{x}\ln(x)$ so $\frac{1}{x\ln(x)} \leq \frac{1}{\sqrt{x}\ln(x)}$ so

$$\int_3^\infty \frac{1}{\sqrt{x}\ln(x)} \geq \int_3^\infty \frac{1}{x\ln(x)}dx = \infty.$$

So this integral diverges.

16. Does $\int_1^\infty e^{-5x\sqrt{x}}dx$ converge?

Solution: For $x \geq 1$, we know that $x\sqrt{x} \geq x$ so $-x\sqrt{x} \leq -x$ and so $e^{-5x\sqrt{x}} \leq e^{-5x}$ and so

$$\int_1^\infty e^{-5x\sqrt{x}}dx \leq \int_1^\infty e^{-5x}dx = \frac{e^{-5}}{5}.$$

So this integral converges.

17. Does $\int_1^\infty \frac{x}{\sqrt{x^2+1}-e^{-x}}dx$ converge?

Solution: We have that $\sqrt{x^2+1}-e^{-x} \leq \sqrt{x^2+1}$ so $\frac{x}{\sqrt{x^2+1-e^{-x}}} \geq \frac{x}{\sqrt{x^2+1}}$ and so

$$\int_1^\infty \frac{x}{\sqrt{x^2+1}-e^{-x}}dx \geq \int_1^\infty \frac{x}{\sqrt{x^2+1}}dx = \infty,$$

so this integral diverges.

18. Does $\int_0^\infty \frac{1}{(1+x^2)^2}dx$ converge?

Solution: We know that $(1+x^2) \geq 1$ and so $(1+x^2)^2 \geq (1+x^2)$ so $\frac{1}{(1+x^2)^2} \leq \frac{1}{1+x^2}$ so

$$\int_0^\infty \frac{1}{(1+x^2)^2}dx \leq \int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

So this integral converges.

19. Does $\int_1^\infty \sqrt{x}e^{-2x}$ converge?

Solution: We have that $\sqrt{x} \leq x$ for $x \geq 1$ and so $\sqrt{x}e^{-2x} \leq xe^{-2x}$ and so

$$\int_1^\infty \sqrt{x}e^{-2x} \leq \int_1^\infty xe^{-2x} = \frac{e^{-2}}{2} + \frac{e^{-2}}{4},$$

so the integral converges.

20. Does $\displaystyle\int_1^\infty \frac{2x + 2xe^{-x}}{1+x^2}dx$ converge?

Solution: We know that $1+e^{-x} \geq 1$ and so $2x(1+e^{-x}) \geq 2x$ and so $\frac{2x+2xe^{-x}}{1+x^2} \geq \frac{2x}{1+x^2}$ and

$$\int_1^\infty \frac{2x + 2xe^{-x}}{1+x^2}dx \geq \int_1^\infty \frac{2x}{1+x^2}dx = \infty.$$

So, the integral diverges.